A NEW BOUNDARY CONDITION SOLVED WITH B.I.E.M.

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1. INTRODUCTION

Among the classical operators of mathematical physics the Laplacian plays an important role due to the number of different situations that can be modelled by it. Because of this a great effort has been made by mathematicians as well as by engineers to master its properties till the point that nearly everything has been said about them from a qualitative viewpoint.

Quantitative results have also been obtained through the use of the new numerical techniques sustained by the computer. Finite element methods and boundary techniques have been successfully applied to engineering problems as can be seen in the technical literature (for instance [1], [2], [3]).

Boundary techniques are especially advantageous in those cases in which the main interest is concentrated on what is happening at the boundary. This situation is very usual in potential problems due to the properties of harmonic functions.

In this paper we intend to show how a boundary condition different from the classical, but physically sound, is introduced without any violence in the discretization frame of the Boundary Integral Equation Method.

The idea will be developed in the context of heat conduction in axisymmetric problems but it is hoped that its extension to other situations is straightforward. After the presentation of the method several examples will show the capabilities of modelling a physical problem.

2. AXISYMMETRIC PROBLEMS

It is well known that problems with axial symmetry can be treated by reducing their dimensionality, for instance the discretization with Finite Elements can be done only in a transverse section. It is clear that this possibility in connection with the reduction, intrinsic to BIEM, will transform an axisymmetric problem into a series of equations on the line defining the boundaries of a typical cross-section.

This has been fully exploited by several researchers (4 , 5) and can be summarised as follows.

The general basic equation is

$$c\,\phi + \int_{\partial\Omega} \phi \frac{\partial \psi}{\partial n}\, ds = \int_{\partial\Omega} \frac{\partial \phi}{\partial n}\, ds \qquad (1)$$

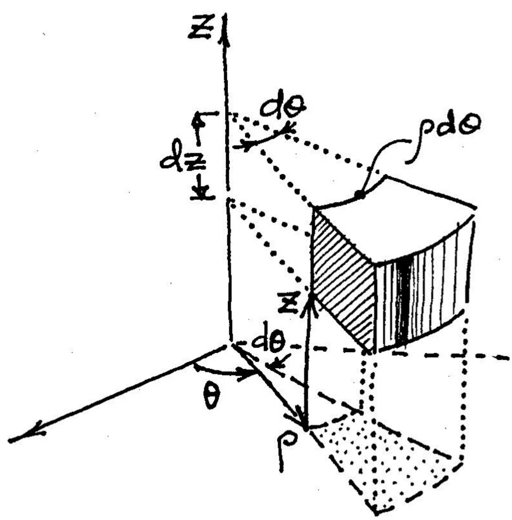


Figure 1

where

$$\psi = \frac{1}{4\pi\rho}$$

is the fundamental solution of Laplace's equation and ρ the distance between the observation point and that in which ψ is "applied".

In polar coordinates (Figure 1)

$$ds = \rho \; d\theta \; dz \tag{2}$$

and taking into account the symmetry of the problem equation (1) can be transformed into

$$c\,\phi + \int_{z_1}^{z_2} \rho \; \phi \; dz \; \frac{\partial \psi_{as}}{\partial n} = \int_{z_1}^{z_2} \rho \; \frac{\partial \phi}{\partial n} \; dz \; \psi_{as} \tag{3}$$

where

$$\psi_{as} = \int_0^{2\pi} \psi \; d\theta = \int_0^{2\pi} \frac{1}{4\pi\rho} \; d\theta = \frac{Q_{-\frac{1}{2}}(\gamma)}{2\,\pi\,(\rho\rho_i)^{\frac{1}{2}}}$$

$$\frac{\partial \psi_{as}}{\partial n} = \int_0^{2\pi} \frac{\partial \psi}{\partial n} \; d\theta = \frac{1}{2\,\pi\,(\rho\rho_i)^{\frac{1}{2}}} \left(- \frac{Q_{-\frac{1}{2}}(\gamma)}{2} + \frac{\rho^2 - \rho_i^2 - (z-z_i)^2}{2\,\rho\rho_i} \right.$$

$$\left. \frac{dQ_{-\frac{1}{2}}(\gamma)}{d\gamma} \right) \rho, n + \left(\frac{z-z_i}{\rho_i} \; \frac{dQ_{-\frac{1}{2}}(\gamma)}{d\gamma} \right) z,n \,\}$$

(4)

where

$$\gamma = 1 + \frac{(\rho - \rho_i)^2 + (z - z_i)^2}{2\,\rho\rho_i}$$

and

$Q_{-\frac{1}{2}}$ is the Legendre function of the second kind.

Discretization of equation (3) is done in the usual way (see for instance ref. [6]) by defining the evolution of inside "boundary elements" as function of ϕ and its derivatives at the "nodes".

It is possible to define as many equations of type (3) as nodes and the imposition of the boundary conditions allows the solution of the problem.

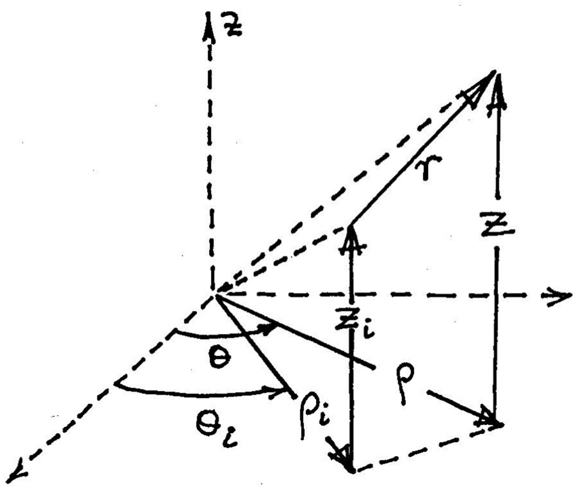


Figure 2

As is well known the boundary conditions for a classical properly posed problems are of the following types

a) Dirichlet condition

$$\phi = \phi_o$$

b) Newmann condition

$$\frac{\partial \phi}{\partial n} = q_o$$

c) Newton (or Robin) condition

$$\alpha \phi + \beta \frac{\partial \phi}{\partial n} = 0 \qquad (6)$$

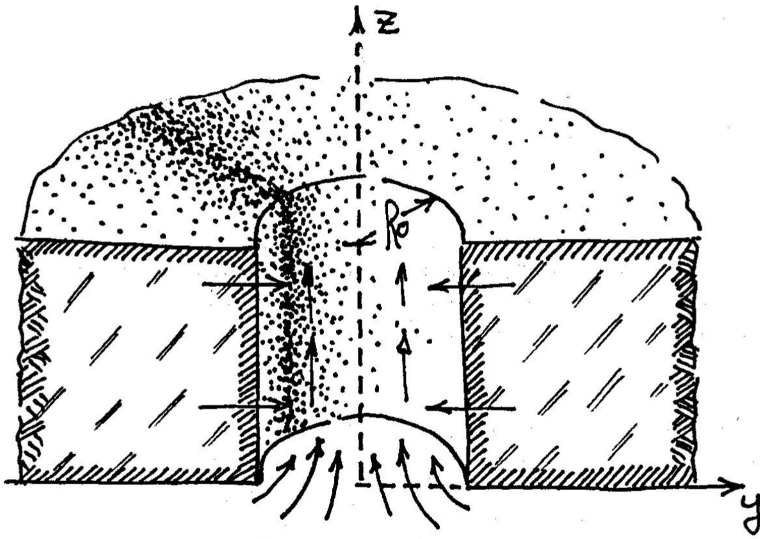


Figure 3

The discussion of the different possibilities for imposing these conditions can be seen for instance in [4], [6], [7].

In what follows we describe a different condition and how to treat it in the BIEM context.

3. SPECIAL BOUNDARY CONDITIONS

Imagine an axisymmetric situation as that sketched in Figure 3.

A fluid receives heat through the walls of a circular channel and eliminates it by mass-flow. The problem can be imagined in the heat propagation field with a field equation of the type

$$\frac{k_z}{k_y} \frac{\partial^2 T}{\partial z^2} + \frac{1}{y} \frac{\partial}{\partial y} \left(y \frac{\partial T}{\partial y} \right) = 0 \qquad (7)$$

where k_x and k_y are the conductivities of the solid body in orthogonal directions. As is well known [8], this equation can be reduced to an isotropic situation by a simple change of scale in the direction x.

The flux of heat in a tube element is

$$k_y \, q(R_o) \;\; 2\,\pi \, R_o \, dz$$

$$q(R_o) = \left(\frac{\partial T}{\partial y}\right)_{y=R_o} \qquad (8)$$

T is the temperature in the tube-fluid interface, that is it is assumed that there is no convection and that the tube is so small that the temperature is constante for every z. If it is a steady-state flow, (8) can be equated to the transported heat.

Let m be the transport velocity (kg/seg) and c_p the specific heat of the fluid. When the unit mass is moved from z to $z + dz$ its quantity of heat is modified by

$$c_p \,(T + \frac{\partial T}{\partial z} \; dz) - c_p \, T = c_p \frac{\partial T}{\partial z} \; dz \qquad (9)$$

so that

$$k_y \, q(R_o) \; 2\,\pi \, R_o \; dz = m \; c_p \frac{\partial T}{\partial z} \; dz \qquad (10)$$

The last equation can be written as a condition on the boundary

$$\frac{\partial T}{\partial y} = \mu \, \frac{\partial T}{\partial z} \qquad (11)$$

with

$$\mu = \frac{m \;\; c_p}{2\,\pi \, k_y \;\; R_o}$$

Equation (11), which is the new boundary condition, relates the normal derivative to the tangential one.

In order to develop its numerical treatment let us remember the right hand side of equation (3)

$$RHS = \int_{z_1}^{z_2} \rho \, \frac{\partial \phi}{\partial n} \; dz \;\; \psi_{as} \qquad (12)$$

The parts of the boundary with the classical conditions are treated as usual. Along the tube and after substitution of (11) in (12), and integrating by parts

$$RHS = \int_{z_1}^{z_2} \psi_{as} \; \mu \frac{\partial T}{\partial z} \rho \, dz = \mu \left(\left(\psi_{as} \; \rho \; \phi \right)_{z_1}^{z_2} - \int_{z_1}^{z_2} \phi \frac{\partial}{\partial z} (\psi_{as} \rho) \, dz \right) \tag{13}$$

At this moment the discretization used to model ϕ can be introduced

$$\phi = \underset{\sim}{N} \; \underset{\sim}{\phi}^{e} \tag{14}$$

and the last integral in (13) can be expressed as

$$\left(\int \underset{\sim}{N} \frac{\partial}{\partial z} (\psi_{as} \; \rho) \, dz \right) \; \underset{\sim}{\phi}^{e} \tag{15}$$

It is clearly advantageous to use constant elements in which case

$$RHS = \mu \left(\left(\psi_{as} \; \rho \; \phi \right)_{z_1}^{z_2} - \sum_{j=1}^{M} \phi_j \int \frac{\partial}{\partial z} (\psi_{as} \, \rho) \, dz \right) =$$

$$= \mu \left(\left(\psi_{as} \; \rho \; \phi \right)_{z_1}^{z_2} - \sum_{j=1}^{M} \phi_j \left(\psi_{as} \; \rho \right)_{j_1}^{j_2} \right) \tag{16}$$

Clearly z_1 and z_2 indicate the limits of the zone where the new conditions are applied and (j_1, j_2) to the points defining the j element.

As the form of the terms is the same it is possible to include the first two into the sum without problem.

4. EXAMPLES

As an example of the application of the previous method imagine the problem of Figure (4a). A thick-wall tube with internal diameter $_{o}$ and external R is formed with an orthotropic material with conductivities k_y, k_z while the extreme bases of the cylinder are maintained at temperatures T_o and T_H.

Equation (7) can be transformed into

$$\frac{k_x R^2}{k_y H^2} \frac{\partial^2 \theta}{\partial \xi^2} + \frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \theta}{\partial \eta} \right) = 0 \qquad (17)$$

by the rule

$$\theta = \frac{T - T_o}{T_H - T_o}$$

$$\xi = \frac{z}{H} \qquad (18)$$

$$\eta = \frac{y}{R}$$

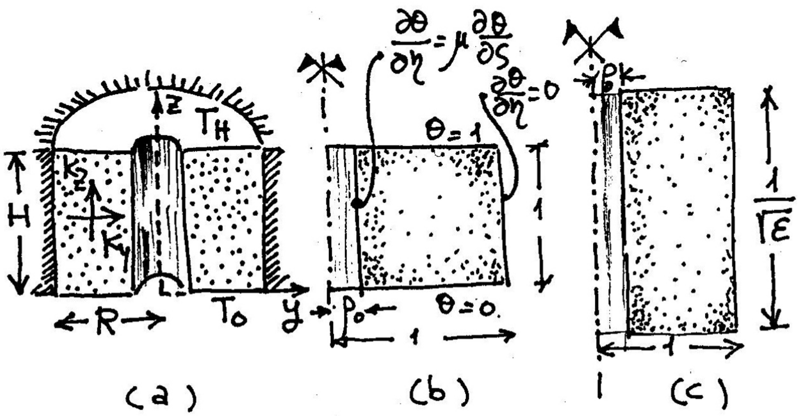


Figure 4

over the domain of Figure (4b).

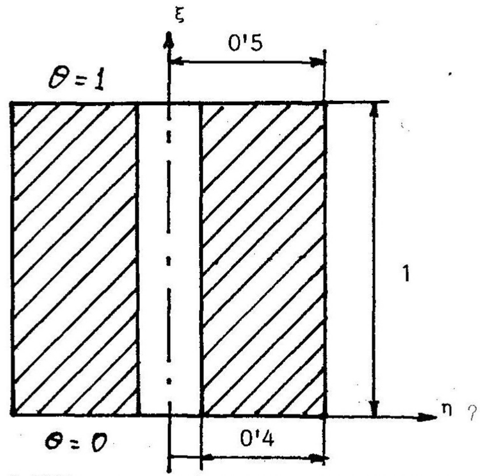
Condition (11) is also modified accordingly. The BIEM can be applied directly to the problem but it is also possible to change again the geometry by doing

$$\xi' = \frac{1}{\sqrt{\frac{k_x R^2}{k_y H^2}}} \qquad \xi = \frac{H}{R}\sqrt{\frac{k_y}{k_x}}\ \xi \qquad (19)$$

obtaining finally

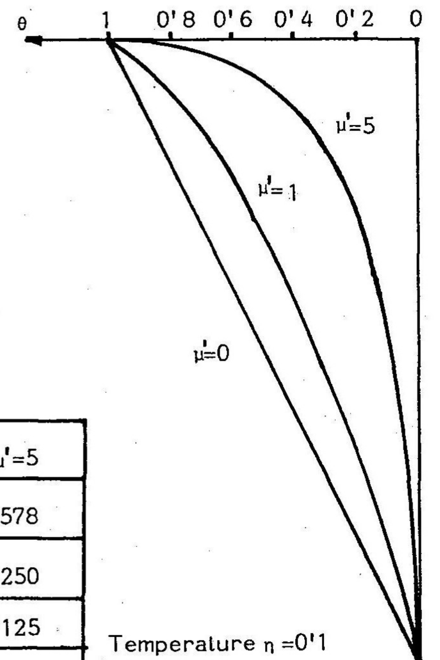
$$\frac{\partial^2 \theta}{\partial \xi'} + \frac{1}{\eta} \frac{\partial}{\partial \eta} \left(\eta \frac{\partial \theta}{\partial \eta} \right) = 0 \qquad (20)$$

which is the differential equation of an axisymmetric potential problem



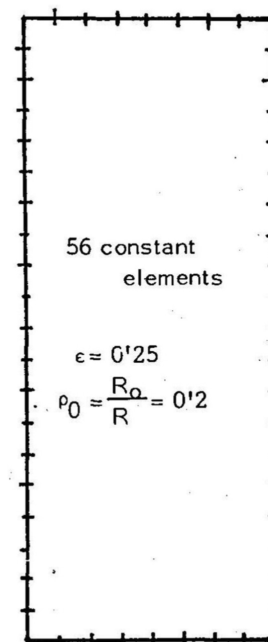
ξ	Analytical $\mu' = 0$	$\mu' = 0$	$\mu' = 1$	$\mu' = 5$
0'975	0'975	0'9765	0'921	0'578
0'775	0'775	0'7753	0'603	0'250
0'575	0'575	0'5751	0'393	0'125
0'375	0'375	0'3748	0'218	0'053
0'175	0'175	0'1746	0'077	0'013

Temperature at $\eta = 0'1$



Temperature $\eta = 0'1$

$$\left(\frac{\partial \theta}{\partial \eta} = \mu' \frac{\partial \theta}{\partial \xi'} \right)$$



defined over the domain of Figure (4c) where

$$\sqrt{\varepsilon} = \frac{R}{H}\sqrt{\frac{k_x}{k_y}} \qquad \mu' = \frac{\mu}{\sqrt{\varepsilon}} \tag{21}$$

Figure 5 shows the discretization for a particular case as well as the evolution of the temperature in the channel.

5. CONCLUSIONS

The main objective of the paper has been to show the ease with which BIEM can include non-classical boundary conditions.

The problem posed was described elsewhere [9] and corresponds to the physcial situation that arises in the neck of vapour cooled shield (VCS) dewars where the evaporation of a cryogenic is used to refrigerate the thermal isolation. More details can be seen in (9).

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